

A Method for Solving Higher Order Homogeneous Ordinary Differential Equations with Non-Constant Coefficients

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Abstract

Higher order differential equations as a field of mathematics has gained importance with regards to the increasing mathematical modeling and penetration of technical and scientific processes. This paper constitutes a presentation of some established old methods with emphasis on their limitations as well as the development of a newly proposed method for solving linear higher order ordinary differential equations with non-constant coefficients. This alternative solution eliminates the need for the commonly employed searching/guessing techniques of finding one linearly independent solution in order to obtain the other linearly independent solutions for the type of higher order differential equations considered.

Keywords: differential equation, reduction of order, non-constant coefficients, higher order, homogeneous equations

INTRODUCTION

“Ordinary differential equations” is a wide mathematical discipline which is closely related to both pure mathematical research and real engineering world applications. Most mathematical formulations of physical laws are described in terms of ordinary and partial differential equations, and this has been a great motivation for their study in the past. In the 20th century the extremely fast development of Science led to applications in the fields of chemistry, biology, medicine, population dynamics, genetic engineering, economy, social sciences and others, as well (Canada et al., 2004; Agarwal et al., 2002; Bogнар and Dosly, 2003). All these disciplines promoted to higher level and new discoveries were made with the help of this kind of mathematical modeling. At the same time, real world problems have been and continue to be a great inspiration for pure mathematics, particularly concerning ordinary differential equations: they led to new mathematical models and challenged mathematicians and research engineers to look for new methods to solve them (Canada and Ruiz, 2003; Cecchi et al., 2001; Fan et al., 2002; Cabada et al., 2000).

It should also be mentioned that an extremely fast development of computer sciences took place in the last three decades: mathematicians have been provided with a tool which had not been available before. This fact encouraged scientists to formulate more complex mathematical models which, in the past, could hardly be resolved or even understood. Even if computers rarely permit a rigorous treatment of a problem, they are a very useful tool to get concrete numerical results or to make interesting

numerical experiments. In the field of ordinary differential equations this phenomenon led more and more mathematicians to the study of more complex differential equations [Shampine and Gordon, 1975; Bui and Bui, 1979; Aynlar and Tiryaki, 2002; Dosly, 2003]. The work at hand pretty well reflects the “state of the art” in the theory of ordinary differential equations with particular emphasis on some established old methods as well as a newly proposed method to solve higher order ordinary differential equations with non-constant coefficients of the general form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \quad (1)$$

BASIC NOTATIONS

A differential equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = F(x) \quad (2)$$

is called an n -th order linear differential equation. Here F and the coefficients a_i are functions of x which are supposed to be continuous in a certain interval. If $a_0, a_1, \dots, a_n$ are constants, we call it a differential equation with constant coefficients. If $F = 0$, then the linear differential equation is homogeneous, and if $F \neq 0$, then it is inhomogeneous. Solution of homogeneous n -th order linear differential equation with constant coefficients has been well established, for example, via the root of the corresponding characteristics equation (Hartman, 1982). In the case of the inhomogeneous non-constant coefficient n -th order linear ordinary differential equations, the solutions of the corresponding homogeneous equation are employed as the basis to determine the solution of the

inhomogeneous differential equation by the method of undetermined coefficients (Bronshtein et al., 2005). This is done in accordance with the Superposition Principle, which states that if $y_1(x)$ and $y_2(x)$ are two solutions of the differential equation (2) for different right-hand sides $F_1(x)$ and $F_2(x)$, then their sum $y = y_1 + y_2$ is a solution of the same differential equation with the right-hand side $F = F_1 + F_2$. From this observation it follows that to get the general solution of an inhomogeneous differential equation it is sufficient to add any particular solution of the inhomogeneous differential equation to the general solution of the corresponding homogeneous differential equation.

The solution becomes more complicated however when $a_0, a_1, \dots, a_n$ are functions of x , i.e. in the case of non-constant coefficient linear ordinary differential equations whose solutions would be further developed in this paper.

Reduction of Order

One of the most important solution methods for n -th order linear differential equations is the substitution of certain variables in order to obtain a simpler differential equation, especially one of lower order. The simplest case is when $y^{(n)}$ is an explicit function of x , wherein the general solution is obtained by n repeated integrations. Different more complicated cases can be highlighted:

➤ When x does not appear explicitly:

$$f(y, y', \dots, y^{(n)}) = 0 \quad (3)$$

By making appropriate substitutions as follows into equation (3):

$$\frac{dy}{dx} = p; \frac{d^2y}{dx^2} = p \frac{dp}{dy}; \frac{d^3y}{dx^3} = p \left(\left(\frac{dp}{dy} \right)^2 + p \frac{d^2p}{dy^2} \right); \dots \quad (4)$$

the order of the differential equation will be reduced from n to $(n-1)$.

➤ When y does not appear explicitly:

$$f(x, y', \dots, y^{(n)}) = 0 \quad (5)$$

The order of the differential equation can be reduced from n to $(n-1)$ by making appropriate substitutions as follows:

$$\frac{dy}{dx} = p; \frac{d^2y}{dx^2} = \frac{dp}{dx}; \frac{d^3y}{dx^3} = \left(\frac{d^2p}{dx^2} \right); \dots \quad (6)$$

this is made easier if the first i -th derivatives are missing in the differential equation, then it would be necessary to substitute $p = y^{(i+1)}$.

➤ When $f(x, y, y', \dots, y^{(n)}) = 0$ is a homogeneous function in $y, y', \dots, y^{(n)}$:

$$f(x, y, y', \dots, y^{(n)}) = 0 \quad (7)$$

The order of the differential equation could be lowered from n to $(n-1)$ by transforming the differential equation (7), making use of the following substitutions:

$$y' = \frac{dy}{dx} = zy \Rightarrow y = e^{\int z dx}; \frac{d^2y}{dx^2} = y \frac{dz}{dx} + zy^2; \dots \quad (8)$$

Fundamental Set of Solutions

A system of n solutions $y_1, y_2, \dots, y_n$ of a homogeneous linear differential equation (1) is called a *fundamental set of solutions* if these functions are *linearly independent* on the considered interval, i.e., their linear combination $C_1y_1 + C_2y_2 + \dots + C_ny_n$ is not identically zero for any system of values $C_1, C_2, \dots, C_n$ except for the values $C_1 = C_2 = \dots = C_n = 0$. The solutions $y_1, y_2, \dots, y_n$ of a linear homogeneous differential equation form a fundamental set of solutions on the considered interval if and only if their *Wronskian determinant*, W , is non-zero:

$$0 \neq W = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \quad (9)$$

If the solutions $y_1, y_2, \dots, y_n$ form a fundamental set of solutions of the differential equation (1), then the general solution of the linear homogeneous differential equation is given as

$$y = C_1y_1 + C_2y_2 + \dots + C_ny_n \quad (10)$$

It therefore implies that a linear n -th order homogeneous differential equation has exactly n *linearly independent* solutions on an interval, where the coefficient functions $a_i(x)$ are continuous.

Lowering the Order if y_1 is Known

If we know a particular solution $y_1(x)$ of a homogeneous differential equation, then by assumption and appropriation of the substitution $y_2(x) = y_1(x) \cdot u(x)$ into (10), and subsequently into (1), we can determine further solutions $y_2(x), \dots, y_n(x)$ progressively from a homogeneous linear differential equation of a reduced order $(n-1), (n-2), \dots$ for $z = u'(x)$, etc.

For the sake of illustration of the above concepts, we will restrict our further investigations to the case of real constants $a_0 \dots a_n$ in the linear homogeneous higher order differential equation with non-constant coefficient of the type:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = 0.$$

For instance, let us consider a second order linear homogeneous differential equation with non-constant coefficient of the following form:

$$ax^2 y'' + bxy' + cy = 0 \quad (11)$$

Now if a particular linearly independent solution to the differential equation (11) is $y_1(x)$ and we seek the second linearly independent solution as $y_2(x) = y_1(x) \cdot u(x)$, it follows that:

$$y_2' = y_1' u + u' y_1; \text{ and } y_2'' = y_1'' u + 2u' y_1' + u y_1'' \quad (12)$$

By plugging in $y_2(x) = y_1(x) \cdot u(x)$ together with equations (12) into (11), after appropriate mathematical manipulations we obtain the following:

$$u(ax^2y_1'' + bxy_1' + cy_1) + ax^2u''y_1 + u'(2ax^2y_1' + bxy_1) = 0 \quad (13)$$

Now since $y_1(x)$ is a particular linearly independent solution to the differential equation (11), it will satisfy equation (11). If this is taken note of in equation (13), and further substitutions

$$z(x) = u'(x); z' = u''(x) \quad (14)$$

are made, equation (13) could be reduced to a first order linear differential equation in $z = u'(x)$:

$$ax^2y_1z' + z(2ax^2y_1' + bxy_1) = 0 \quad (15)$$

Dividing through by ax^2y_1 the homogeneous equation (15) could be expressed in standard form:

$$z' + z\left(\frac{2y_1'}{y_1} + \frac{b}{ax}\right) = 0 \quad (16)$$

The integrating factor for equation (16) could be evaluated thus:

$$\exp \int \left(\frac{2y_1'}{y_1} + \frac{b}{ax} \right) dx = \exp(\ln y_1^2) \cdot \exp\left(\ln x^{b/a}\right) = y_1^2 x^{b/a} \quad (17)$$

We get the solution of equation (16) thus:

$$z = u' = \frac{du}{dx} = \frac{C}{y_1^2 x^{b/a}} \Rightarrow u = \int \left(\frac{C}{y_1^2 x^{b/a}} \right) dx \quad (18)$$

Subsequently, the second linearly independent solution $y_2(x)$ will be

$$y_2 = uy_1 = y_1 \int \left(\frac{C}{y_1^2 x^{b/a}} \right) dx \quad (19)$$

And finally the general solution to second order linear homogeneous differential equation (11) with non-constant coefficient in accordance with equation (10) will take the form:

$$y = C_1y_1 + C_2y_2 = y_1 \left(C_1 + C_2 \int \left(\frac{C}{y_1^2 x^{b/a}} \right) dx \right) \quad (20)$$

Alternative Solution If $Y_1, Y_2, \dots, Y_n$ Are Unknown

Existing solution methods of higher order linear ordinary differential equations with non-constant coefficients, for instance by variation of parameters or the just considered reduction of order depend on the availability of a known linearly independent solution, which serves as the basis for obtaining the other fundamental sets of solution, first for the homogeneous equation, then followed by the assumption of analogies between the homogeneous and the inhomogeneous higher order differential equations. This is however based on guessing and/or trial and error methods to obtain a known linearly independent solution. We present an alternative solution method with no previously given independent solution. This method is suitable and tested for all linear homogeneous n -order differential equations with non-constant coefficients of the type:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 xy' + a_0 y = 0,$$

illustrated still with the aid of the example given in equation (11). We could seek a particular solution $y_1(x)$ of the differential equation (11) in the form $y_1(x) = x^{k_1}$, where k_1 is a non-zero constant. Thus,

$$y_1 = x^k \Rightarrow y_1' = kx^{(k-1)} \text{ and } y_1'' = k(k-1)x^{(k-2)} \quad (21)$$

Plugging in y_1, y_1' , and y_1'' from (21) into (11) and subsequent algebraic transformations yield the following:

$$k(k-1)ax^2x^{k-2} + b k x x^{(k-1)} + c x^k = 0 \Rightarrow ak^2 + (b-a)k + c = 0 \quad (22)$$

The roots of the defining quadratic equation (22), k_1 and k_2 are given by:

$$(23) \quad k_1 = \frac{(a-b) + \sqrt{(b-a)^2 - 4ac}}{2a} \text{ and } k_2 = \frac{(a-b) - \sqrt{(b-a)^2 - 4ac}}{2a}$$

It is obvious from equation (23) that the roots of the resulting defining equation (22) could be same or different and then we could get two linearly independent solutions of the differential equation (11). Several cases could be highlighted for real, imaginary roots:

Case 1: $k_1 \neq k_2$

We can be convinced that the two linearly independent solutions $y_1(x) = x^{k_1}$, and $y_2(x) = x^{k_2}$, since their Wronkian determinant from equation (9) will be non-zero as long as $k_1 \neq k_2$. The general solution of the differential equation (11) will take the form:

$$y = C_1y_1 + C_2y_2 = C_1x^{k_1} + C_2x^{k_2} \quad (24)$$

Case 2: $k_1 = -k_2$, when $a = b$

A special case could be distinguished when the roots k_1, k_2 could be expressed as $\pm k$. Substitution of the equity $a = b$ into equation (23) gives

$$k_1 = \sqrt{-c/a} \text{ and } k_2 = -\sqrt{-c/a} \quad (25)$$

The general solution of the differential equation (11) in the case is expressible as follows:

$$y = C_1x^{\sqrt{-c/a}} + C_2x^{-\sqrt{-c/a}} = x^{\sqrt{-c/a}} \left(C_1 + C_2x^{-2\sqrt{-c/a}} \right) \quad (26)$$

Case 3: $k_1 = k_2 = k$

Here we obtain only one linearly independent solution $y_1(x) = x^k$ and we can get a second solution $y_2(x)$ with help of equation (19). By plugging in the appropriate values into equation (19):

$$y_2 = y_1 \int \left(\frac{C}{y_1^2 x^{b/a}} \right) dx = x^k \int \left(\frac{C}{x^{2k} x^{b/a}} \right) dx = x^k \left(\frac{Cx^{(1-2k-\frac{b}{a})}}{(1-2k-\frac{b}{a})} + C_0 \right) \quad (27)$$

It is noteworthy that $2k + \frac{b}{a} \neq 1$ to avoid division by zero in equation (27). For the special case, when $2k + \frac{b}{a} = 1$ this would be taken note of before integration is performed and it can easily be shown that the general solution to the differential equation (11) will take a simplified form:

$$y = x^k (C_0 + C \ln x) \quad (28)$$

CONCLUSION

This study focuses on the review and development of the techniques for solving linear higher order ordinary differential equations with non-constant coefficients. The generally established methods have been presented while highlighting their limitations and thereafter employed to develop a new method. The significance of the newly proposed method lies in the elimination of the need for commonly employed searching/guessing techniques of finding one linearly independent solution in order to obtain the other linearly independent solutions for the type of higher order differential equations considered. This solution could be applicable to the development of special solutions of engineering models for various types of real problems. Mathematicians and scientists interested in the recent results and methods in the theory and applications of ordinary differential equations will find the paper useful.

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